Veronese transform, and Castelnuovo-Mumford regularity of modules

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Abstract [1]

Veronese rings, Segre embeddings or more generally Segre-Veronese embeddings are very important rings in Algebraic Geometry. In this paper we present an original, elementary way to compute the Hilbert-Poincare series of these rings, as a consequence we compute their Castelnuovo-Mumford regularity, and also the leading term of the $h-$vector. Moreover, we can compute the Castelnuovo-Mumford regularity of the n-Veronese Module of any finitely generated Cohen-Macaulay graded module.

1 Introduction

Veronese rings, Segre embeddings or more generally Segre-Veronese embeddings are very important rings in Algebraic Geometry. It is well known that these rings are arithmetically Cohen-Macaulay, hence their Hilbert-Poincaré series can be written $P_R(t) = \frac{Q_R(t)}{(1-t)^{\dim R}}$, where $Q_R(t)$ is a polynomial on t with $Q_R(1) \neq 0$ having positive integer coefficients, the sequence of the coefficients of $Q_R(t)$ is also called the $h-$vector of R. The degree of $Q_R(t)$ is the Castelnuovo-Mumford regularity (c.f.[7][Chapter 4]), and the leading term of $Q_R(t)$ is the highest graded Betti number of R. By using very original and elementary methods we are able to compute the leading term of $Q_R(t)$. Our results allows to compute the Castelnuovo-Mumford regularity of the n Veronese Module of any finitely generated Cohen-Macaulay graded module, and the rings called of Veronese type.

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Note that this result can be proved easily by using local cohomology, but our purpose is to give a very elementary proof.

Our main results improves partially [1] and [5].

Theorem. *Let consider the Segre-Veronese ring* $R_{\underline{b},\underline{n}}$, $\dim R_{\underline{b},\underline{n}} = b_1 + ... + b_m + 1$. *Let* $P_{R_{\underline{b},\underline{n}}}(t) = \frac{Q_{R_{\underline{b},\underline{n}}}}{(1-t)^{\dim R_{\underline{b},\underline{n}}}}$ *be the Hilbert-Poincaré series of* $R_{\underline{b},\underline{n}}$, *with* $Q_{R_{\underline{b},\underline{n}}} = h_0 + h_1 t + ... + h_{r_{\underline{b},\underline{n}}} t^{r_{\underline{b},\underline{n}}}$, *where* $r_{\underline{b},\underline{n}}$ *is the Castelnuovo-Mumford regularity of* $R_{\underline{b},\underline{n}}$. *We set* $\alpha_{\underline{b},\underline{n}} = \dim R_{\underline{b},\underline{n}} - r_{\underline{b},\underline{n}}$. *After a permutation of* $b_1, ..., b_m$, *we can assume that, for all* $i = 1, ..., m$, $\lceil \frac{b_1+1}{n_1} \rceil > \frac{b_i}{n_i}$. *Then*

$$\alpha_{\underline{b},\underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil \quad ,$$

and the highest Betti number of $R_{\underline{b},\underline{n}}$ *is*

$$\beta_{r_{\underline{b},\underline{n}}} = h_{r_{\underline{b},\underline{n}}} = \binom{n_1\alpha_{\underline{b},\underline{n}} - 1}{b_1} ... \binom{n_m\alpha_{\underline{b},\underline{n}} - 1}{b_m}$$

Theorem. *Fix integers* $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$. *Let* $(a_l)_{l\in\mathbb{Z}}$ *be a sequence of complex numbers, such that* $a_l = 0$ *for* $l << 0$, *set :*

$$f(t) = \sum_{l\in\mathbb{Z}} a_l t^l, \quad f^{<n,\tau>}(t) = \sum_{l\in\mathbb{Z}} a_{nl+\tau} t^l.$$

If $f(t) = \frac{h(t)}{(1-t)^d}$ *with* $h(t) \in \mathbb{C}[t, t^{-1}]$ *then* $f^{<n,\tau>}(t) = \frac{h^{<n>}(t)}{(1-t)^d}$ *with* $h^{<n>}(t) \in \mathbb{C}[t, t^{-1}]$ *such that:*

- $\deg h^{<n,\tau>}(t) \leq d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$,
- *If all the coefficients of* $h(t)$ *are positive real numbers then* $\deg h^{<n,\tau>}(t) = d - \lceil \frac{d - \deg h(t) + \tau}{n} \rceil$,
- *If* $\deg h(t) = d$ *then* $\deg h^{<n>}(t) = d$.

2 Preliminaries on toric rings and Hilbert-Poincaré series

Let $S = K[x_0, ..., x_s, x_0^{-1}, ..., x_s^{-1}]$ be a Laurent polynomial ring over a field K on a finite set of variables. For any finite set $\mathcal{M}$ of monomials in S, let $K[\mathcal{M}] \subset S$ be the subring of S generated by the set

$\mathcal{M}$. It is the toric ring defined by the semigroup generated by $\mathcal{M}$. In what follows we consider the special case where $S = K[x_0, ..., x_s]$ is a polynomial ring over the field K and all the monomials in $\mathcal{M}$ are of the same degree.

Example 2.1. *Let $S = K[x_0, ..., x_b] \oplus_{l \in \mathbb{N}} S_l$, and $\mathcal{M} = \{x_0^{\alpha_0} ... x_b^{\alpha_b} \mid \alpha_0 + ... + \alpha_b = n\}$. So that*

$$R_{b,n} = K[\mathcal{M}] = \oplus_{l \in \mathbb{N}} S_{nl}.$$

This toric ring is known as the $n-$Veronese embedding of S.

Example 2.2. *More generally, let $X_1, ..., X_m$, m sets of independent disjoint variables, with $\mathrm{Card}(X_i) = b_i + 1$. Let $S_i = K[X_i]$ for $i = 1, ..., m$, $S = K[X_1 \cup X_2 \cup X_m]$, and $\mathcal{M} = \{x_1 x_2 ... x_m \mid x_i \in X_i\}$. So that*

$$R_{b_1,...,b_m} = K[\mathcal{M}] = \oplus_{l \in \mathbb{N}} (S_1)_l \otimes ... \otimes (S_m)_l.$$

This toric ring is known as the Segre embedding of the m polynomial rings $S_1, ..., S_m$.

Example 2.3. *Let $X_1, ..., X_m$, sets of independent disjoint variables such that $X_i = \{x_{i,0}, ..., x_{i,b_i}\}$, $S_i = K[X_i]$ for $i = 1, ..., m$, and $n_1, ..., n_m \in \mathbb{N}$, Let $S = K[X_1 \cup X_2 \cup X_m]$, and*

$$\mathcal{M} = \{\underline{x}_1^{\alpha_1} ... \underline{x}_m^{\alpha_m} \mid\mid \alpha_i \mid= n_i\},$$

where $\alpha_i = (\alpha_{i,0}, ..., \alpha_{i,b_i})$, $\underline{x}_i^{\alpha_i} = x_{i,0}^{\alpha_{i,0}} ... x_{i,b_i}^{\alpha_{i,b_i}}$, and $\mid \alpha_i \mid= \alpha_{i,0} + ... + \alpha_{i,b_i}$ The Segre-Veronese embedding:

$$R_{\underline{b},\underline{n}} = K[\mathcal{M}] = \oplus_{l \in \mathbb{N}} (S_1)_{n_1 l} \otimes ... \otimes (S_m)_{n_m l},$$

where $\underline{b} = (b_1, ..., b_m), \underline{n} = (n_1, ..., n_m)$.

Let $S = K[x_0, ..., x_s]$ be a polynomial ring over the field K, graded by the standard graduation, that is $\deg x_i = 1$, for all i. Let $R := S/I$, where $I \subset S$ is a graded ideal, let $M = \oplus_{l \in \mathbb{Z}} M_l$ be a finitely generated graded R-module, hence M is also a $S-$module. the Hilbert-function of M is defined by $H_M(l) = \dim_K M_l$, for all $l \in \mathbb{Z}$, and the Hilbert-Poincaré series of M:

$$P_M(t) = \sum_{l \in \mathbb{Z}} H_M(l) t^l.$$

It is well known that

$$P_M(t) = \frac{Q_M(t)}{(1-t)^{\dim M}} \quad ,$$

where $Q_M(t)$ is a Laurent polynomial on t, t^{-1} with $Q_M(1) \neq 0$. Moreover if M is a Cohen-Macaulay $S-$module, all the coefficients of $Q_M(t)$ are natural integers,and the Castelnuovo-Mumford regularity of M is the degree of $Q_M(t)$. For more details on Hilbert-Poincare series see [10], [4][Chapter 4], [7][Chapter 4].

Theorem 2.4. *(Hilbert's Theorem) let $M = \oplus_{l\in\mathbb{Z}} M_l$ be a finitely generated graded S-module. There exists a polynomial with integer coefficients $\Phi_{H_M}(l)$ such that $H_M(l) = \Phi_{H_M}(l)$, for l large enough. Moreover the leading term of $\Phi_{H_M}(l)$ can be written as : $\frac{\deg(M)}{d!} l^d$, where $d+1$ is the dimension of M and $\deg(M)$ the degree or multiplicity of M.*

Remark 2.5. *The postulation number of the Hilbert function is the biggest integer l such that $H_M(l) \neq \Phi_{H_M}(l)$. It is well known , ([10], [4][Chapter 4]), that the postulation number equals the degree of the rational fraction defining the Poincaré series.*

Remark 2.6. *We recall that binomial coefficients can be defined in a more general setting than natural numbers, indeed for $k \in \mathbb{N}$, binomial coefficients are polynomial functions in the variable n. More precisely:*

(1) *If $k = 0$ then let $\binom{n}{0} = 1$, for all $n \in \mathbb{C}$.*
(2) *If $k > 0$ then let $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$, for all $n \in \mathbb{C}$.*

Note that for all $n \in \mathbb{C}$, $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$ and if $n \in \mathbb{N}, n < k$, then $\binom{n}{k} = 0$.

Example 2.7. *Let $S = K[x_0, ..., x_b]$, be a polynomial ring. Then*

$$H_S(l) = \begin{cases} \binom{l+b}{b} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases} \quad , \qquad P_S(t) = \frac{1}{(1-t)^{b+1}}.$$

Note that in fact $\forall l \geq -b, \quad H_S(l) = \binom{l+b}{b}$ and $0 = H_S(-b-1) \neq \binom{-b-1+b}{b} = (-1)^b$, so the postulation number of S is $-(b+1)$.

Example 2.8. *Let $S = K[x_0, ..., x_b]$, $\mathcal{M} = \{x_0^{\alpha_0}...x_b^{\alpha_b} \mid \alpha_0+...+\alpha_b = n\}$, and $R_{b,n} = K[\mathcal{M}]$ the $n-$Veronese embedding. Then*

$$H_{R_{b,n}}(l) = H_S(nl) = \begin{cases} \binom{nl+b}{b} & \text{if } l \geq 0 \\ 0 & \text{if } l < 0 \end{cases}.$$

Note that $\binom{nl+b}{b} = \frac{(nl+1)(nl+2)...(nl+b)}{b!}$ is a polynomial on l with leading term $\frac{n^b l^b}{b!}$, so that $\deg(R_{b,n}) = n^b, \dim R_{b,n} = b+1$. Note also that $\forall l > -\lceil \frac{b+1}{n} \rceil$, $H_{R_{b,n}}(l) = \binom{nl+b}{b}$ and $0 = H_{R_{b,n}}(-\lceil \frac{b+1}{n} \rceil) \neq \binom{-\lceil \frac{b+1}{n} \rceil n+b}{b} = (-1)^b \binom{\lceil \frac{b+1}{n} \rceil n-1}{b}$, so the postulation number of $R_{b,n}$ is $-\lceil \frac{b+1}{n} \rceil$. More generally the postulation number of $R_{b,n}[\tau]$ is $\lceil \frac{b+1+\tau}{n} \rceil$.

3 Veronese of generating series

In a recent paper [2], Brenti and Walker prove that taking the $n-$ Veronese transform of the h polynomial is a linear function, in this section we improve this result giving a elementary proof of the fact that taking the shifted $n-$ Veronese transform of the h polynomial is a linear function on h.

Let recall the following fact:

Theorem 3.1. *Let $(a_l)_{l\in\mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l << 0$, set : $f(t) = \sum_{l\in\mathbb{Z}} a_l t^l$, TFAE:*

- *There exists $h(t) \in \mathbb{C}[t, t^{-1}]$ and a natural integer d such that $f(t) = \frac{h(t)}{(1-t)^d}$.*
- *There exists $\Phi(t) \in \mathbb{C}[t, t^{-1}]$ of degree $d-1$ with leading coefficient $e_0/(d-1)!$, such that $\Phi(l) = a_l$ for l large enough.*

Moreover $h(1) = e_0$.

Definition 3.2. *Fix integers $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$. Let $(a_l)_{l\in\mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l << 0$, set :*

$$f(t) = \sum_{l\in\mathbb{Z}} a_l t^l, \quad f^{<n,\tau>}(t) = \sum_{l\in\mathbb{Z}} a_{nl+\tau} t^l.$$

By the Theorem 3.1 if $f(t) = \frac{h(t)}{(1-t)^d}$ with $h(t) \in \mathbb{C}[t, t^{-1}]$ then $f^{<n,\tau>}(t) = \frac{h^{<n,\tau>}(t)}{(1-t)^d}$ with $h^{<n,\tau>}(t) \in \mathbb{C}[t, t^{-1}]$.

Let us introduce some notations. To any non zero polynomial $h(t) = h_\sigma t^\sigma + ... + h_0 + h_1 t + ... + h_s t^s \in \mathbb{C}[t, t^{-1}]$ we associate the h-vector $\vec{h} = (..., 0, h_\sigma, ..., h_s, 0, ...)$, and we set $\deg \overrightarrow{h} = \deg h(t)$. For $j \in \mathbb{Z}$, let $\overrightarrow{\varepsilon_j}$ be the h-vector of the polynomial t^j. Let denote by $[t^k]h(t)$ the coefficient of t^k in the polynomial $h(t)$. For any $i, j \in \mathbb{Z}$ define $\mathcal{D}_{i,j}$ by

$$\mathcal{D}_{i,j} = [t^{in-j}](\frac{(1-t^n)^d}{(1-t)^d}) = [t^{in-j}]((1+t+...t^{n-1})^d).$$

Note that

$$\mathcal{D}_{i,j} = \mathrm{Card}\{(x_1, ..., x_d) \in \mathbb{N}^d \mid \forall l, x_l \leq n-1; x_1 + ... + x_d = in - j\}.$$

Finally let $\mathcal{D}[\sigma, \tau]$ be the infinite square matrix $\mathcal{D}[\sigma, \tau] = (\mathcal{D}_{i+\sigma, j+\tau})$. For $\sigma = \tau = 0$ we write $\mathcal{D}$ instead $\mathcal{D}[0, 0]$. We can give some properties of the numbers $\mathcal{D}_{i,j}$.

Lemma 3.3. *Let $i, j, k \in \mathbb{Z}$, we have:*

- $\mathcal{D}_{i,j} = 0$ *if either $in - j < 0$ or $in - j > d(n-1)$.*
- *For any i, j, $\mathcal{D}_{i,j} = \mathcal{D}_{d-i,d-j}$. That is $\mathcal{D}$ is symmetrical around the point $(d/2, d/2)$.*
- *For $0 \leq k \leq n-1$, $\mathcal{D}_{d,d+k} = \binom{k+d-1}{d-1}$.*
- $\mathcal{D}_{1,0} = \binom{n+d-1}{d-1} - d$, *and for $1 \leq k \leq n$, $\mathcal{D}_{1,k} = \binom{n-k+d-1}{d-1}$.*
- *For any integers q, k, $\mathcal{D}_{d+q,nq+k} = \mathcal{D}_{d,k}$.*
- *For any i, let $d - i = nq - k$ with $q = \lceil \frac{d-i}{n} \rceil, 0 \leq k < n$, then*

$$\mathcal{D}_{d-\lceil \frac{d-i}{n} \rceil, i} = \binom{k+d-1}{d-1} = \binom{n\lceil \frac{d-i}{n} \rceil + i - 1}{d-1}.$$

Proof. The first claim is trivial. In order to prove the other claims, let remark that the map $(x_1, ..., x_d) \mapsto (y_1, ..., y_d)$, where $y_l = (n-1) - x_l$ for $l = 1, ..., d$, establishes a bijection between

$$\{(x_1, ..., x_d) \in \mathbb{N}^d \mid x_l \leq n-1 \text{ for } l = 1, ..., d; \;\; x_1 + ... + x_d = in - j\}$$

and

$$\{(y_1, ..., y_d) \in \mathbb{N}^d \mid y_l \leq n-1 \text{ for } l = 1, ..., d; \;\; y_1 + ... + y_d = (d-i)n - (d-j)\}.$$

The third claim follows from the second claim, because if $0 \leq k \leq n-1$, then the sets

$$\{(x_1, ..., x_d) \in \mathbb{N}^d \mid x_l \leq n-1 \text{ for } l = 1, ..., d; \;\; x_1+...+x_d = dn-d-k\}$$

and

$$\{(y_1, ..., y_d) \in \mathbb{N}^d \mid \quad y_1 + ... + y_d = k\}$$

are in bijection.

The fourth claim follows trivially from the previous items.

The fifth claim follows from the equality: $(d+q)n - (nq+k) = dn - k$.

Finally the sixth claim follows from the third claim, since, if $d - i = nq - k$ with $0 \leq k < n$, then $(d-q)n - i = dn - (d+k)$, hence $\mathcal{D}_{d-q,i} = \mathcal{D}_{d,d+k}$, and $n\lceil \frac{d-i}{n} \rceil + i - 1 = k + d - 1$.

$\square$

Remark 3.4. *With the notations introduced in 3.2, it is clear that* $f^{<n,kn+\tau>}(t) = t^{-k} f^{<n,\tau>}(t)$, *which implies* $h^{<n,kn+\tau>}(t) = t^{-k}h^{<n,\tau>}(t)$ *for any integer numbers* k, τ.

The following Theorem improves and gives a simpler proof of [2, Theorem 1.1]:

Theorem 3.5. *Fix integers* $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$. *Let* $(a_l)_{l\in\mathbb{Z}}$ *be a sequence of complex numbers, such that* $a_l = 0$ *for* $l << 0$, *set :*

$$f(t) = \sum_{l\in\mathbb{Z}} a_l t^l = \frac{h(t)}{(1-t)^d},$$

$$f^{<n,\tau>}(t) = \sum_{l\in\mathbb{Z}} a_{nl+\tau} t^l = \frac{h^{<n,\tau>}(t)}{(1-t)^d},$$

where $h(t), h^{<n,\tau>}(t) \in \mathbb{C}[t, t^{-1}]$. *Then*

$$\text{for any } \tau \in \mathbb{Z}, \overrightarrow{h^{<n,kn+\tau>}} = \mathcal{D}[-k, -\tau]\vec{h}.$$

Proof. Because the Remark 3.4 we have to compute $h^{<n,\tau>}(t)$ only for $0 \leq \tau \leq n-1$. The following formula is clear:

$$f^{<n,0>}(t^n) + tf^{<n,1>}(t^n) + ... + t^{n-1}f^{<n,n-1>}(t^n) = f(t),$$

hence

$$\frac{h^{<n,0>}(t^n) + th^{<n,1>}(t^n) + ... + t^{n-1}h^{<n,n-1>}(t^n)}{(1-t^n)^d} = \frac{h(t)}{(1-t)^d},$$

and

$$h^{<n,0>}(t^n) + th^{<n,1>}(t^n) + ... + t^{n-1}h^{<n,n-1>}(t^n) = h(t)\frac{(1-t^n)^d}{(1-t)^d},$$

$t^\tau h^{<n,\tau>}(t^n)$ equals the sum of all the terms $A_\beta t^\beta$ of $h(t)\frac{(1-t^n)^d}{(1-t)^d}$ with $\beta \equiv \tau \mod n$. In particular $h^{<n,\tau>}(t)$ is a linear function of $h(t)$. So it is enough to compute $h^{<n,\tau>}(t)$ for the canonical basis $\{\varepsilon_j := t^j\ , j \in \mathbb{Z}\}$ of $\mathbb{C}[t, t^{-1}]$. We have

$$[t^i](h^{<n,\tau>}(t)) = [t^{ni+\tau}](h(t)\frac{(1-t^n)^d}{(1-t)^d}),$$

hence

$$\forall j \in \mathbb{Z}; [t^i](\varepsilon_j^{<n,\tau>}(t)) = [t^{ni+\tau}](t^j)\frac{(1-t^n)^d}{(1-t)^d} = [t^{ni+\tau-j}](\frac{(1-t^n)^d}{(1-t)^d})$$

which proves our statement. $\square$

Corollary 3.6. *Fix an integer $d \in \mathbb{N}^*$. For $j \in \mathbb{Z}$, let $\overrightarrow{\varepsilon_j}$ be the h-vector of the polynomial t^j. Then for any $n \in \mathbb{N}^*$, we have $\deg \overrightarrow{\varepsilon_j^{<n>}} = d - \lceil \frac{d-j}{n} \rceil$. Moreover the leading coefficient of $\overrightarrow{\varepsilon_j^{<n>}}$ is $\binom{n\lceil \frac{d-j}{n}\rceil + j - 1}{d-1}$.*

Proof. Let remark that the set of $t^j, j \in \mathbb{Z}$ is the canonical basis of $\mathbb{C}[t, t^{-1}]$. We have by Theorem 3.5 that $\mathcal{D}\overrightarrow{\varepsilon_j} = \overrightarrow{\varepsilon_j^{<n>}}$, hence $\overrightarrow{\varepsilon_j^{<n>}}$ is the j column vector of $\mathcal{D}$. By the Example 2.8, we have that $\deg \overrightarrow{\varepsilon_j^{<n>}} = d - \lceil \frac{d-j}{n} \rceil$.

The last claim follows from the Lemma 3.3. Indeed for any $j \in \mathbb{Z}$, we have $\mathcal{D}_{d-\lceil \frac{d-j}{n}\rceil, j} = \binom{n\lceil \frac{d-j}{n}\rceil + j - 1}{d-1}$. This proves that the leading coefficient of $\overrightarrow{\varepsilon_j^{<n>}}$ is $\binom{n\lceil \frac{d-j}{n}\rceil + j - 1}{d-1}$. $\square$

Example 3.7. *Let $d = 2$ and $n \in \mathbb{N}^*$, we can describe the matrix $\mathcal{D}$*

ij	*-(n+1)*	...	*-1*	*0*	*1*	*2*	*3*	...	*n*	*n+1*	*n+2*	...	*2n*
-1	*2*	...	*0*	*0*	*0*	*0*	*0*	...	*0*	*0*	*0*	....	*0*
0	*n-2*	...	*2*	*1*	*0*	*0*	*0*	...	*0*	*0*	*0*	...	*0*
1	*0*	...	*n-2*	*n-1*	*n*	*n-1*	*n-2*	...	*1*	*0*	*0*	...	*0*
2	*0*	...	*0*	*0*	*0*	*1*	*2*	...	*n-1*	*n*	*n-1*	...	*1*
3	*0*	...	*0*	*0*	*0*	*0*	*0*	...	*0*	*0*	*1*	...	*n-1*

Theorem 3.8. *Fix integers $d, n \in \mathbb{N}^*, \tau \in \mathbb{Z}$. Let $(a_l)_{l\in\mathbb{Z}}$ be a sequence of complex numbers, such that $a_l = 0$ for $l << 0$, set :*

$$f(t) = \sum_{l\in\mathbb{Z}} a_l t^l, \quad f^{<n,\tau>}(t) = \sum_{l\in\mathbb{Z}} a_{nl+\tau} t^l.$$

If $f(t) = \frac{h(t)}{(1-t)^d}$ with $h(t) \in \mathbb{C}[t, t^{-1}]$ then $f^{<n,\tau>}(t) = \frac{h^{<n>}(t)}{(1-t)^d}$ with $h^{<n>}(t) \in \mathbb{C}[t, t^{-1}]$ such that:

- $\deg h^{<n,\tau>}(t) \leq d - \lceil \frac{d-\deg h(t)+\tau}{n} \rceil$,
- *If all the coefficients of $h(t)$ are positive real numbers then* $\deg h^{<n,\tau>}(t) = d - \lceil \frac{d-\deg h(t)+\tau}{n} \rceil$,
- *If $\deg h(t) = d$ then $\deg h^{<n>}(t) = d$.*

Proof. Let $f(t) = \sum_{l\in\mathbb{Z}} a_l t^l = \frac{h(t)}{(1-t)^d}$, where $h(t) \in \mathbb{C}[t, t^{-1}]$ $h(t) = \gamma_\sigma t^\sigma + ... + \gamma_s t^s$ with $\deg h(t) = s, \gamma_s \neq 0$. It follows that $\overrightarrow{h} = \sum_{l=\sigma}^{s} \gamma_l \overrightarrow{\varepsilon_l}$. We multiply this relation on the left by $\mathcal{D}[-\tau]$, so Theorem 3.5 implies $\overrightarrow{h^{<n,\tau>}} = \sum_{l=\sigma}^{s} \gamma_l \overrightarrow{\varepsilon_{l-\tau}^{<n>}}$. Since $\deg \overrightarrow{\varepsilon_{\sigma-\tau}^{<n>}} \leq \deg \overrightarrow{\varepsilon_{\sigma-\tau+1}^{<n>}} \leq ... \leq \deg \overrightarrow{\varepsilon_{s-\tau}^{<n>}}$, we have, $\deg \overrightarrow{h^{<n,\tau>}} \leq \deg \overrightarrow{\varepsilon_{s-\tau}^{<n>}}$. It is clear that if all the coefficients of $h(t)$ are positive real numbers then $\deg \overrightarrow{h^{<n,\tau>}} = \deg \overrightarrow{\varepsilon_{s-\tau}^{<n>}}$.

In the special case $s = d$, we have seen that for $0 \leq l \leq d-1$ and any $n \in \mathbb{N}^*$, $\deg \overrightarrow{\varepsilon_l^{<n>}} = d - \lceil \frac{d-l}{n} \rceil \leq d-1$, and $\deg \overrightarrow{\varepsilon_d^{<n>}} = d$, which implies $\deg \overrightarrow{h^{<n>}} = d$. □

Theorem 3.9. *Let $n \in \mathbb{N}^*$, S be a standard graded polynomial ring, $M = \oplus_{l\in\mathbb{Z}} M_l$ be a finitely generated Cohen-Macaulay graded S-module of dimension $d \geq 1$, and $M^{<n>} = \oplus_{l\in\mathbb{Z}} M_{nl}$. Let $\dfrac{Q(t)}{(1-t)^d}$ be the Hilbert-Poincaré series of M, where $Q(t) = \gamma_\sigma t^\sigma + ... + \gamma_s t^s \in \mathbb{C}[t, t^{-1}]$ is the $h-$polynomial of M, with $\operatorname{reg}(M) = \deg Q(t) = s$. Then*

- $\operatorname{reg} M^{<n>} = d - \lceil \frac{d-\operatorname{reg} M}{n} \rceil$. *Moreover by taking the sum over all index l such that $\lceil \frac{d-l}{n} \rceil = \lceil \frac{d-\operatorname{reg} M}{n} \rceil$, we will get the leading coefficient of $Q^{<n>}(t)$:*

$$\sum_{l \ | \ \lceil \frac{d-l}{n} \rceil = \lceil \frac{d-\operatorname{reg} M}{n} \rceil} \gamma_l \binom{n\lceil \frac{d-l}{n} \rceil + l - 1}{d-1}.$$

- *If* $\operatorname{reg} M \leq d-1$ *and* $n \geq d$ *then* $\operatorname{reg} M^{<n>} = d-1$*, and the leading coefficient of* $Q^{<n>}(t)$ *is*

$$\sum_{l=0}^{d-1} \gamma_l \binom{n\lceil \frac{d-l}{n} \rceil + l - 1}{d-1}.$$

- *If* $n > \operatorname{reg} M \geq d$ *then* $\operatorname{reg} M^{<n>} = d$*, and the leading coefficient of* $Q^{<n>}(t)$ *is*

$$\sum_{l=d}^{\operatorname{reg} M} \gamma_l \binom{l-1}{d-1}.$$

Proof. We have $\overrightarrow{Q} = \sum_{l=\sigma}^{s} \gamma_l \overrightarrow{\varepsilon_l}$. We multiply this relation on the left by $\mathcal{D}$, so Theorem 3.5 implies that for any $n \in \mathbb{N}^*$, $\overrightarrow{Q^{<n>}} = \sum_{l=\sigma}^{s} \gamma_l \overrightarrow{\varepsilon_l^{<n>}}$. Since $\gamma_l \geq 0$ for all l, $\gamma_s > 0$, and $\deg \overrightarrow{\varepsilon_\sigma^{<n>}} \leq \deg \overrightarrow{\varepsilon_{\sigma+1}^{<n>}} \leq \ldots \leq \deg \overrightarrow{\varepsilon_s^{<n>}}$, we have, $\deg \overrightarrow{Q^{<n>}} = \deg \overrightarrow{\varepsilon_s^{<n>}} = d - \lceil \frac{d - \operatorname{reg} M}{n} \rceil$, this number is $\operatorname{reg}(M^{<n>})$ since $M^{<n>}$ is a Cohen-Macaulay $S-$module. The computation of the leading coefficient of $Q^{<n>}(t)$ is immediate from Lemma 3.6. □

4 h-vector of the Segre-Veronese embedding.

The next Theorem improves partially [1] and [5].

Theorem 4.1. *Let consider the Segre-Veronese ring* $R_{\underline{b},\underline{n}}$*,* $\dim R_{\underline{b},\underline{n}} = b_1 + \ldots + b_m + 1$*. Let* $P_{R_{\underline{b},\underline{n}}}(t) = \dfrac{Q_{R_{\underline{b},\underline{n}}}(t)}{(1-t)^{\dim R_{\underline{b},\underline{n}}}}$ *be the Hilbert-Poincaré series of* $R_{\underline{b},\underline{n}}$*, with* $Q_{R_{\underline{b},\underline{n}}}(t) = h_0 + h_1 t + \ldots + h_{r_{\underline{b},\underline{n}}} t^{r_{\underline{b},\underline{n}}}$*, where* $r_{\underline{b},\underline{n}} = \deg Q_{R_{\underline{b},\underline{n}}}(t)$ *is the Castelnuovo-Mumford regularity of* $R_{\underline{b},\underline{n}}$*. We set* $\alpha_{\underline{b},\underline{n}} = \dim R_{\underline{b},\underline{n}} - r_{\underline{b},\underline{n}}$*. After a permutation of* $b_1, \ldots, b_m$*, we can assume that* $\lceil \frac{b_1+1}{n_1} \rceil > \frac{b_i}{n_i}$ $\forall i$*, then*

$$\alpha_{\underline{b},\underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil \quad r_{\underline{b},\underline{n}} = (b_1 + \ldots + b_m + 1) - \lceil \frac{b_1+1}{n_1} \rceil,$$

and the highest Betti number of $R_{\underline{b},\underline{n}}$ *is*

$$\beta_{r_{\underline{b},\underline{n}}} = h_{r_{\underline{b},\underline{n}}} = \binom{n_1 \alpha_{\underline{b},\underline{n}} - 1}{b_1} \ldots \binom{n_m \alpha_{\underline{b},\underline{n}} - 1}{b_m}$$

Proof. The proof is by double induction on m and b_m. The case $m=1$ is given by the Example 2.8 and Corollary 3.6, so we can assume $m \geq 2$. We have that $\lceil \frac{b_1+1}{n_1} \rceil > \frac{b_m}{n_m} > \frac{b_m-1}{n_m}$, so by induction hypothesis the theorem is true for $R_{\underline{b}-\epsilon_m,\underline{n}}$, where $\underline{b}-\epsilon_m = (b_1,...,b_{m-1},b_m-1)$. On the other hand the Hilbert function of $R_{\underline{b},\underline{n}}$ is $H_{R_{\underline{b},\underline{n}}}(l) = \binom{n_1 l+b_1}{b_1}...\binom{n_m l+b_m}{b_m}$, so

$$H_{R_{\underline{b},\underline{n}}}(l) = (1+\frac{n_m}{b_m}l)H_{R_{\underline{b}-\epsilon_m,\underline{n}}}(l). \tag{1}$$

Let $P_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t) = \dfrac{Q_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t)}{(1-t)^{b_1+...+b_m}}$ be the Hilbert-Poincaré series of $R_{\underline{b}-\epsilon_m,\underline{n}}$, where $Q_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t) = h_0+h_1t+...+h_{r_{\underline{b}-\epsilon_m,\underline{n}}}t^{r_{\underline{b}-\epsilon_m,\underline{n}}}$, with $h_{r_{\underline{b}-\epsilon_m,\underline{n}}} \neq 0$. In order to avoid any confusion we also set : $P_{R_{\underline{b},\underline{n}}}(t) = \dfrac{Q_{R_{\underline{b},\underline{n}}}(t)}{(1-t)^{b_1+...+b_m+1}}$ be the Hilbert-Poincaré series of $R_{\underline{b},\underline{n}}$, where

$$Q_{R_{\underline{b},\underline{n}}}(t) = \hat{h}_0+...+\hat{h}_{r_{\underline{b},\underline{n}}}t^{r_{\underline{b},\underline{n}}},$$

with $\hat{h}_{r_{\underline{b},\underline{n}}} \neq 0$.

Let $\beta = \frac{n_m}{b_m}$, by simple calculations from (1) we get:

$$P_{R_{\underline{b},\underline{n}}}(t) = P_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t) + \beta t P'_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t). \tag{2}$$

Hence $\dim R_{\underline{b},\underline{n}} = \dim R_{\underline{b}-\epsilon_m,\underline{n}} + 1$, and

$$Q_{R_{\underline{b},\underline{n}}}(t) = Q_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t) + t[Q_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t)(\beta R_{\underline{b}-\epsilon_m,\underline{n}}-1) + \beta Q'_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t) - \beta t Q'_{R_{\underline{b}-\epsilon_m,\underline{n}}}(t)],$$

note that $Q_{R_{\underline{b},\underline{n}}}(1) = \beta \dim R_{\underline{b}-\epsilon_m,\underline{n}} Q_{R_{\underline{b}-\epsilon_m,\underline{n}}}(1) \neq 0$.

In particular we have $r_{\underline{b},\underline{n}} \leq r_{\underline{b}-\epsilon_m,\underline{n}}+1$ and for all $k=0,...,r_{\underline{b}-\epsilon_m,\underline{n}}+1$ we have

$$\hat{h}_k = h_{k-1}(\frac{n_m}{b_m}\dim R_{\underline{b}-\epsilon_m,\underline{n}} - (k-1)\frac{n_m}{b_m} - 1) + h_k(k\frac{n_m}{b_m}+1). \tag{3}$$

By induction hypothesis we have $\alpha_{\underline{b}-\epsilon_m,\underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil \neq \frac{b_m}{n_m}$, so we put $k = r_{\underline{b}-\epsilon_m,\underline{n}}+1$ in equality (3), and we get:

$$\hat{h}_{r_{\underline{b}-\epsilon_m,\underline{n}}+1} = h_{r_{\underline{b}-\epsilon_m,\underline{n}}}(\frac{n_m\alpha_{\underline{b}-\epsilon_m,\underline{n}} - b_m}{b_m}) \neq 0.$$

Hence $\hat{h}_{r_{\underline{b}-\epsilon_m,\underline{n}}+1}$ is the leading coefficient of $Q_{R_{\underline{b},\underline{n}}}$ and $r_{\underline{b},\underline{n}} = r_{\underline{b}-\epsilon_m,\underline{n}}+1$ and $\alpha_{\underline{b},\underline{n}} = \alpha_{\underline{b}-\epsilon_m,\underline{n}} = \lceil \frac{b_1+1}{n_1} \rceil$. By induction hypothesis

$$h_{r_{\underline{b}-\epsilon_m,\underline{n}}} = \binom{n_1\alpha_{\underline{b},\underline{n}}-1}{b_1}...\binom{n_{m-1}\alpha_{\underline{b},\underline{n}}-1}{b_{m-1}}\binom{n_m\alpha_{\underline{b},\underline{n}}-1}{b_m}$$

so that

$$\hat{h}_{r_{\underline{b},\underline{n}}} = \binom{n_1\alpha_{\underline{b},\underline{n}}-1}{b_1}\dots\binom{n_{m-1}\alpha_{\underline{b},\underline{n}}-1}{b_{m-1}}\binom{n_m\alpha_{\underline{b},\underline{n}}-1}{b_m-1}(\frac{n_m\alpha_{\underline{b},\underline{n}}-b_m}{b_m})$$
$$= \binom{n_1\alpha_{\underline{b},\underline{n}}-1}{b_1}\dots\binom{n_{m-1}\alpha_{\underline{b},\underline{n}}-1}{b_{m-1}}\binom{n_m\alpha_{\underline{b},\underline{n}}-1}{b_m}.$$

□

5 Rings of Veronese type

Let $b, n \in \mathbb{N}^*, \underline{a} = (a_0, ..., a_b) \in \mathbb{N}^{b+1}$ such that $1 \leq a_i \leq n, a_0 + ... + a_b > n$, and $\mathcal{M}_{b,n,\underline{a}}$ be the set of monomials of the polynomial ring $K[x_0, ..., x_b]$:

$$\mathcal{M}_{b,n,\underline{a}} = \{x_0^{\alpha_0}...x_b^{\alpha_b} \mid \quad \alpha_0 + ... + \alpha_b = n, \alpha_i \leq a_i, \forall i = 0, ..., b\}.$$

Let denote by $R_{b,n,\underline{a}}$ the toric subring of $K[x_0, ..., x_b]$ generated by $\mathcal{M}_{b,n,\underline{a}}$. It is well known that $R_{b,n,\underline{a}}$ is a Cohen-Macaulay ring. Let $\mathcal{S}$ be the collection of subsets of $\{0, ..., b\}$ such that : $S \in \mathcal{S}$ if and only if $S \subset \{0, ..., b\}$, and $\Sigma S := \sum_{i \in S} a_i < n$.

Theorem 5.1. *([9]) With the above notations the Hilbert-function of $R_{b,n,\underline{a}}$ is*

$$\forall l \geq 0; \quad H_{b,n,\underline{a}}(l) = \sum_{S \in \mathcal{S}} (-1)^{|S|} \binom{l(n - \Sigma S) - \mid S \mid + b}{b}$$

We have $\dim(R_{b,n,\underline{a}}) = b + 1$, and its degree or multiplicity is

$$\deg(R_{b,n,\underline{a}}) = \sum_{S \in \mathcal{S}} (-1)^{|S|} (n - \Sigma S)^b.$$

Our aim is to study the Hilbert-Poincaré series of $R_{b,n,\underline{a}}$:

$$P_{R_{b,n,\underline{a}}} = \sum_{S \in \mathcal{S}} (-1)^{|S|} \sum_{l \geq 0} \binom{l(n - \Sigma S) - \mid S \mid + b}{b} t^l.$$

The following Corollary follows immediately from 3.6.

Corollary 5.2. *For any $S \in \mathcal{S}$ and $k \in \mathbb{N}^*$, we have:*

$$\sum_{l \geq 0} \binom{kl - \mid S \mid + b}{b} t^l = \frac{Q_{S,k}(t)}{(1-t)^{b+1}},$$

where $Q_{S,k}(t)$ is a polynomial with $Q_{S,k}(1) \neq 0$, with leading term

$$\binom{k\alpha_{S,k}+ \mid S \mid -1}{b} t^{b+1-\alpha_{S,k}},$$

with $\alpha_{S,k} = \lceil \frac{b+1-|S|}{k} \rceil$.

The following theorem is immediate from 5.1 and Corollary 5.2. It improves the description of the Hilbert Poincaré series given in [9].

Theorem 5.3. *With the above notations, let $\mathcal{S}$ be the collection of subsets of $\{0,...,b\}$ such that : $S \in \mathcal{S}$ if and only if $S \subset \{0,...,b\}$, and $\Sigma S := \sum_{i \in S} a_i < n$. Then we can write the Hilbert-Poincaré series of $R_{b,n,\underline{a}}$:*

$$P_{R_{b,n,\underline{a}}} = \frac{Q_{b,n,\underline{a}}(t)}{(1-t)^{b+1}},$$

with $Q_{b,n,\underline{a}}(t) = \sum_{S \in \mathcal{S}} (-1)^{|S|} Q_{S,n}(t)$, where $Q_{S,n}(t)$ is a polynomial with $Q_{S,n}(1) \neq 0$, with leading term

$$\binom{(n-\Sigma S)\alpha_{S,n-\Sigma S}+ \mid S \mid -1}{b} t^{b+1-\alpha_{S,n-\Sigma S}},$$

where $\alpha_{S,n-\Sigma S} = \lceil \frac{b+1-|S|}{n-\Sigma S} \rceil$.

Part one of the following corollary improves [9][Cor. 2.12].

Corollary 5.4. *With the above notations:*

1. $\mathrm{reg}(R_{b,n,\underline{a}}) \leq b+1-\lceil \frac{b+1}{n} \rceil$, *and the equality is true if and only if*

$$\sum_{S \in \mathcal{S}, \alpha_{S,n-\Sigma S} = \lceil \frac{b+1}{n} \rceil} (-1)^{|S|} \binom{(n-\Sigma S)\alpha_{S,n-\Sigma S}+ \mid S \mid -1}{b} \neq 0$$

2. *If $b+1 > n^2$ then $\mathrm{reg}(R_{b,n,\underline{a}}) = b+1-\lceil \frac{b+1}{n} \rceil$. Moreover the leading term of $Q_{b,n,\underline{a}}(t)$ is $\binom{n\lceil \frac{b+1}{n} \rceil -1}{b} t^{b+1-\lceil \frac{b+1}{n} \rceil}$,*

Proof. 1. It is enough to prove that $\min_{S \in \mathcal{S}} \lceil \frac{b+1-|S|}{n-\Sigma S} \rceil = \lceil \frac{b+1}{n} \rceil$. We consider two cases,

- if $b+1 < n$ then $\lceil \frac{b+1}{n} \rceil = 1 \leq \lceil \frac{b+1-|S|}{n-\Sigma S} \rceil, \forall S \in \mathcal{S}$.

- If $b+1 \geq n$, then

$$\frac{b+1}{n} \leq \frac{b+1-\mid S\mid}{n-\Sigma S} \Leftrightarrow (b+1)(n-\Sigma S) \leq n(b+1-\mid S\mid) \Leftrightarrow (b+1)\Sigma S \geq n\mid S\mid,$$

this is true since by hypothesis $\frac{b+1}{n} \geq 1 \geq \frac{|S|}{\Sigma S}$.

2. Let $b+1 > n^2$ and $S \neq \emptyset$. By definition $\lceil \frac{b+1}{n} \rceil$ is the integer q such that $b+1 = qn - r$, with $0 \leq r < n$ and $q \geq n+1$. We have

$$b+1-\mid S\mid = qn - r - \mid S\mid = q(n-\Sigma S) - r - \mid S\mid + q\Sigma S,$$

and $q\Sigma S - \mid S\mid \geq (n+1)\Sigma S - \mid S\mid \geq n\Sigma S > r$, so that $q\Sigma S - \mid S\mid - r > 0$, hence $\lceil \frac{b+1-|S|}{n-\Sigma S} \rceil > q = \lceil \frac{b+1}{n} \rceil$.

□

In general leading terms of the alternating sum can cancel, as we can see in the next example.

Example 5.5. *Let consider the ring $R_{4,3,(1,1,1,1,1)}$, the sets S can have 0,1 or 2 elements, and we have: If $S = \emptyset$ then $\alpha_{\emptyset,3} = \lceil \frac{5}{3} \rceil = 2$, if $\mid S\mid = 1$ then $\alpha_{S,3} = \lceil \frac{4}{2} \rceil = 2$, and finally if $\mid S\mid = 2$ then $\alpha_{S,2} = \lceil \frac{3}{1} \rceil = 3$. By using Theorem 5.3 we can write*

$$P_{R_{4,3,(1,1,1,1,1)}} = \frac{Q_0(t) - 5Q_1(t) + 10Q_2(t)}{(1-t)^5},$$

, with $Q_0(t) = 5t^3 + ...; Q_1(t) = t^3 + ...; Q_2(t) = t^2 + ...$ Note that in this case $Q_0(t) - 5Q_1(t) + 10Q_2(t) = h_0 + h_1 t + h_2 t^2$, where $h_0 = 1, h_1 = 5$ and since $h_0 + h_1 + h_2 = \deg(R_{4,3,(1,1,1,1,1)}) = 11$, we get $h_2 = 5$, so that

$$P_{R_{4,3,(1,1,1,1,1)}} = \frac{1+5t+5t^2}{(1-t)^5}.$$

References

[1] Barcanescu, S.; Manolache, N., Betti numbers of Segre-Veronese singularities. Rev. Roum. Math. Pures Appl. 26, 549-565 (1981).

[2] Brenti, F. ; Welker V., The Veronese construction for formal power series and graded algebras Adv. Applied Math., 42 (2009), 545-556.

[3] Brodmann, M.P.; Sharp, R.Y., Local cohomology. An algebraic introduction with geometric applications. Cambridge Studies in Advanced Mathematics. 60. Cambridge: Cambridge University Press. xv, 416 p. (1998).

[4] Bruns, Winfried; Herzog, Jürgen, Cohen-Macaulay rings. Rev. ed. Cambridge Studies in Advanced Mathematics. 39. Cambridge: Cambridge University Press. xiv, 453 p. (1998).

[5] Cox, David A.; Materov, Evgeny, Regularity and Segre-Veronese embeddings. Proc. Am. Math. Soc. 137, No. 6, 1883-1890 (2009).

[6] De Negri, Emanuela; Hibi, Takayuki, Gorenstein algebras of Veronese type. J. Algebra 193, No.2, 629-639 (1997).

[7] Eisenbud, David, The geometry of syzygies. A second course in commutative algebra and algebraic geometry. Graduate Texts in Mathematics 229. New York, NY: Springer. xvi, 243 p. (2005).

[8] Goto, Shiro and Watanabe, Keiichi, On graded rings, I. J. Math. Soc. Japan Volume 30, Number 2 (1978), 179-213.

[9] Katzman, Mordechai, The Hilbert series of algebras of the Veronese type. Commun. Algebra 33, No. 4, 1141-1146 (2005).

[10] Morales M., *Fonctions de Hilbert, genre géométrique d'une singularité quasi-homogène Cohen-Macaulay*. CRAS Paris, t.301, série A n^o 14 **(1985)**.